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# MULTIPLE SCATTERING OF ACOUSTICAL WAVES

by

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### Abstract

The general theory of the multiple scattering of acoustical waves by a random distribution of isotropic point scatterers is considered. Configurational averages are taken of the equations of multiple scattering and integral equations governing these configurational averages are obtained; the physical consequences of these equations are examined in detail. A complete theoretical picture is obtained of the propagation of the coherent and incoherent radiation and the connection between the coherent and incoherent contributions to the average sound intensity and current.

## INTRODUCTION

The general theory of the multiple scattering of particles by a random distribution of scatterers has been extensively considered in recent years with particular application to molecular transport in gases, neutron and gamma-ray transport in matter, radiation transfer in stars, and to a number of other physical phenomena. Such problems are usually studied with some variation of the Boltzmann integro-differential equation describing transport processes. This formulation is merely the expression of conservation of particles in phase space; hence the treatment is classical, with no account taken of the wave nature of the particles or photons. Such a theory would be expected to be valid only if the wave length of the particles is much smaller than the average distance of separation between the scatterers.

There are also a large number of problems of multiple scattering in which the wave length is comparable with the average scatterer separation; some examples of this latter type of problem are acoustic wave propagation in bubbly water, elastic wave propagation in irradiated materials, and the scattering of electrons or x-rays by the nuclei of liquids or amorphous solids. Any treatment of these problems must include the reflection, refraction, and interference phenomena that are characteristic of wave problems; hence it must be based on the wave equation, rather than on the simple conservation statement leading to the Boltzmann equation.

The first systematic treatment based on the wave equation was made by Foldy<sup>1</sup>, who considered the multiple scattering of scalar waves

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<sup>1</sup>L. L. Foldy, "The Multiple Scattering of Waves", Phys. Rev. 67, 107-119 (1945).

by a random collection of isotropic point scatterers. Foldy's unique contribution was the introduction of the concept of "configurational" averaging of relevant physical quantities by defining a joint probability distribution for the occurrence of a particular scatterer configuration. By averaging the equations of multiple scattering over the statistical ensemble of scatterer configurations, Foldy was able to derive integral equations governing these configurational averages. This procedure was later generalized by Lax<sup>2</sup> to treat the multiple scattering of quantum-mechanical waves by point scatterers having quite general scattering characteristics. Application of the configurational averaging technique has been used by Twersky<sup>3</sup> to study the scattering and reflection of acoustic waves by a rough surface and by Waterman and Truell<sup>4</sup> as well as Fikioris and Waterman<sup>5</sup> to treat scattering regions having non-vanishing dimensions. Surveys of the entire field of multiple scattering of waves have been published by Twersky<sup>6</sup> and Burke and Twersky<sup>7</sup>.

Through the use of configurational averaging, the multiple scattering problem admits the natural decomposition into the separate

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<sup>2</sup>M. Lax, "Multiple Scattering of Waves", Rev. Mod. Phys. 23, 287-310 (1951).

<sup>3</sup>V. Twersky, "On Scattering and Reflection of Sound by Rough Surfaces", J. Acoust. Soc. Am. 29, 209 - 225, (1957).

<sup>4</sup>P. C. Waterman and R. Truell, "Multiple Scattering of Waves", J. Math. Phys. 2, 512 - 537, (1961).

<sup>5</sup>J. C. Fikioris and P. C. Waterman, "Multiple Scattering of Waves. II "Hole Corrections" in the Scalar Case", J. Math. Phys. 5, 1413-1420, (1964).

<sup>6</sup>V. Twersky, "On Multiple Scattering of Waves", J. Res. Natl. Bur. Std. 64D, 715-730, (1960).

<sup>7</sup>J. E. Burke and V. Twersky, "On Scattering of Waves by Many Bodies", J. Res. Natl. Bur. Std. 68D, 500 - 510, (1964).

consideration of the "coherent" and "incoherent" radiation. The solution of the coherent problem bridges the gap between molecular and continuum physics since it may be shown that the coherent wave  $\langle \psi \rangle$  satisfies the wave equation with a complex propagation constant; thus a collection of discrete scatterers imbedded in a matrix medium may be replaced by a continuous medium whose properties depend, in general, on position. The coherent wave, being governed by a wave equation, displays the phenomena of refraction and specular reflection. The incoherent radiation, on the other hand, arises as a result of the statistical superposition of the scattered waves due to the random nature of the scatterer distribution; it is governed by an equation that is similar to the integral form of the Boltzmann equation describing the transport of particles.

Although there has been a considerable amount of work since Foldy's paper, the emphasis has been on the coherent radiation. Little attention, however, has been given to the incoherent radiation. Waterman and Truell<sup>4</sup> have pointed out that the equations describing the intensity  $\langle |\psi|^2 \rangle$  and the current  $\langle \psi^* \nabla \psi - \psi \nabla \psi^* \rangle$  merit further investigation; however apparently this has not been done on a general basis.

This paper again considers the physical situation studied by Foldy - - the multiple scattering of scalar waves by a random distribution of isotropic point scatterers. We shall examine, in particular, the conservation relations for the intensity and current in order to obtain a unified picture of the propagation of the coherent and incoherent radiation and the connection between the coherent and incoherent contributions to the average intensity and current. In a subsequent paper the solution of specific problems of multiple scattering will be given.

## I. PRELIMINARIES

Let us consider a homogeneous medium capable of sustaining a harmonic scalar field characterized by the complex wave function  $\psi(\vec{r})$  satisfying the reduced wave equation,

$$(\nabla^2 + \kappa_o^2) \psi(\vec{r}) = 0 \quad ; \quad (1)$$

in the case of a dissipative medium the propagation constant  $\kappa_o$  is complex. Throughout the medium are embedded  $N$  isotropic point scatterers which are randomly distributed with respect to location and with respect to a parameter  $R_j$ , e.g. the radius of the scatterer. The scattering characteristics of each scatterer are assumed to be governed by a known complex scattering coefficient function  $g(\vec{r}, R)$ , defined such that if the wave incident on a scatterer located at  $\vec{r}_j$  is  $\psi^j(\vec{r}_j)$ , then the scattered wave is given by

$$\psi_j(\vec{r}) = g(\vec{r}_j, R_j) \psi^j(\vec{r}_j) E(\vec{r}, \vec{r}_j) \quad , \quad (2)$$

$$E(\vec{r}, \vec{r}_j) = \frac{e^{i\kappa_o |\vec{r} - \vec{r}_j|}}{|\vec{r} - \vec{r}_j|} \quad . \quad (3)$$

Scattering and extinction cross sections may be defined for a single scatterer in terms of its scatterer coefficient; it is also convenient to define scattering and extinction cross section densities as the integrals of these cross sections, weighted with the number density distribution  $n(\vec{r}, R)$ , as follows:

$$\Sigma_s(\vec{r}) = \int_0^\infty 4\pi |g(\vec{r}, R)|^2 n(\vec{r}, R) dR \quad , \quad (4)$$

$$\Sigma_e(\vec{r}) = \int_0^\infty \frac{4\pi}{\kappa_o} \text{Im}\{g(\vec{r}, R)\} n(\vec{r}, R) dR + \alpha_o \quad . \quad (5)$$



In the above equation,  $k_0$  denotes the real part of the complex propagation constant  $\kappa_0$ , while  $\alpha_0$  denotes twice the imaginary part; it is noted that the extinction cross section density includes absorption by the matrix medium as well as by the scatterers.

In problems dealing with the multiple scattering of waves by a collection of randomly distributed point scatterers, the exact configuration of a given collection will not be specified; rather, one will only have information concerning the average distribution of the scatterers, or the probability of occurrence of a particular configuration. For example, if the scatterers are statistically independent of one another, then the only information that is available concerning the scatterer distribution is the scatterer number density distribution  $n(\vec{r}, R)$ . For a random distribution of scatterers, the quantities of interest are the average values of relevant physical quantities, taken over all possible scatterer configurations, consistent with the known statistical data concerning the average or probable distribution of the scatterers; such an average is called a configurational average.

In order to establish the probabilistic concepts defining the configurational averages, let us consider a statistical ensemble consisting of a collection of an infinite number of scatterer configurations. The nature of this ensemble may be made precise by specification of its joint probability distribution  $p(\vec{r}_1, R_1; \dots; \vec{r}_N, R_N)$ , defined such that  $p(\vec{r}_1, R_1; \dots; \vec{r}_N, R_N) d\vec{r}_1 dR_1 \dots d\vec{r}_N dR_N$  is the probability of a configuration of  $N$  scatterers having the  $j$ 'th scatterer located in the volume element  $d\vec{r}_j$  about the position  $\vec{r}_j$  and in the interval  $dR_j$  about  $R_j$  for  $j = 1, 2, \dots, N$ . The probability of a configuration having the



$j$ 'th scatterer occupying  $\vec{dr}_j dR_j$ , regardless of the locations or radii of all the other scatterers, may be obtained by integrating over all but the  $j$ 'th coordinates, as follows:

$$p(\vec{r}_j, R_j) \vec{dr}_j dR_j = \iint \dots \overset{j}{\iint} p(\vec{r}_1, R_1; \dots; \vec{r}_N, R_N) \vec{dr}_1 dR_1 \dots \vec{dr}_N dR_N. \quad (6)$$

The superscript indicates the omission of the integrations over the  $j$ 'th coordinates. It is also convenient to introduce the notion of "conditional" probability. The conditional probability for a configuration having the  $j$ 'th scatterer fixed at location  $\vec{r}_j$  and radius  $R_j$  is defined by

$$p_j(\vec{r}_1, R_1; \dots; \vec{r}_N, R_N) = \frac{p(\vec{r}_1, R_1; \dots; \vec{r}_N, R_N)}{p(\vec{r}_j, R_j)}, \quad (7)$$

the superscript here indicating the omission of the  $j$ 'th coordinates within the parenthesis. It will be assumed that the locations and radii of the scatterers are statistically independent. This requires that the joint probability distribution be expressible as the product of the individual probabilities for each scatterer; that is

$$p(\vec{r}_1, R_1; \dots; \vec{r}_N, R_N) = \prod_{j=1}^N p(\vec{r}_j, R_j). \quad (8)$$

Finally, we note that the probability  $p(\vec{r}_j, R_j)$  is simply equal to the average number density  $n(\vec{r}_j, R_j)$  of the scatterers at location  $\vec{r}_j$  per unit interval about  $R_j$ , divided by the total number of scatterers present,

$$p(\vec{r}_j, R_j) = \frac{n(\vec{r}_j, R_j)}{N}. \quad (9)$$

Let us now consider a complex scalar field  $\psi(\vec{r})$ , produced by the multiple scattering of an incident wave by a configuration of scatterers, and indicate its dependence on the location  $\vec{r}_j$  and the radius  $R_j$  of each scatterer, in addition to the field point  $\vec{r}$  where it is observed; the configurational average is then defined as

$$\begin{aligned} \langle \psi(\vec{r}) \rangle = & \iint \dots \iint \psi(\vec{r} | \vec{r}_1, R_1; \dots; \vec{r}_N, R_N) \\ & \times p(\vec{r}_1, R_1; \dots; \vec{r}_N, R_N) d\vec{r}_1 dR_1 \dots d\vec{r}_N dR_N, \end{aligned} \quad (10)$$

the average being taken over the statistical ensemble of scatterer configurations; the configurational averages of other quantities are defined in a similar fashion. The exciting field of the  $j$ 'th scatterer  $\psi^j(\vec{r}_j)$  depends on the locations and radii of all the other scatterers, in addition to  $\vec{r}_j$ . Therefore we shall define the conditional configurational average of this quantity by averaging it over a statistical ensemble of scatterer configurations having the  $j$ 'th scatterer fixed; such an average may be defined in terms of the conditional probability distribution, as follows:

$$\begin{aligned} \langle \psi^j(\vec{r}_j) \rangle_j = & \iint \dots \iint \psi^j(\vec{r}_j | \vec{r}_1, R_1; \dots; \vec{r}_N, R_N) \\ & \times p_j(\vec{r}_1, R_1; \dots; \vec{r}_N, R_N) d\vec{r}_1 dR_1 \dots d\vec{r}_N dR_N. \end{aligned} \quad (11)$$

We have defined the "configurational" average of a physical quantity as an average taken over a statistical ensemble of scatterer configurations. Hence, the average field  $\langle \psi(\vec{r}) \rangle$  represents the average value of a set of simultaneous measurements of  $\psi(\vec{r})$  taken on the distinct members of a collection of similar scatterer systems. If the average deviation of  $\psi(\vec{r})$  from its average value is small, i.e.

$$\langle |\psi(\vec{r}) - \langle \psi(\vec{r}) \rangle|^2 \rangle = \langle |\psi(\vec{r})|^2 \rangle - |\langle \psi(\vec{r}) \rangle|^2 \ll \langle |\psi(\vec{r})|^2 \rangle, \quad (12)$$

then a single measurement of  $\psi(\vec{r})$  on a given system would be expected, with high probability, to be very close to the average value  $\langle \psi(\vec{r}) \rangle$ .

We may also envision a physical situation in which the locations  $\vec{r}_j(t)$  and the radii  $R_j(t)$  of the scatterers in a single configuration are slowly changing with time such that it continuously passes through the various states of the statistical ensemble (we assume that the time required for the configuration to undergo a significant change is much greater than the period of oscillation of the sound field). The field that is produced by the multiple scattering of an incident wave by this system depends on time in addition to the point at which it is observed,

$$\psi(\vec{r}, t) = \psi(\vec{r} | \vec{r}_1(t), R_1(t); \dots; \vec{r}_N(t), R_N(t)) \quad (13)$$

Hence a "time" average of this quantity may be defined by

$$\overline{\psi(\vec{r}, t)} = \frac{1}{T} \int_0^T \psi(\vec{r}, t) dt \quad (14)$$

where  $T$  is the length of time over which the average is taken. This average value has little practical significance and would be difficult to measure unless the mean square deviation is small, i. e.

$$\frac{1}{T} \int_0^T |\psi(\vec{r}, t) - \overline{\psi(\vec{r}, t)}|^2 dt = \overline{|\psi(\vec{r}, t)|^2} - |\overline{\psi(\vec{r}, t)}|^2 \ll \overline{|\psi(\vec{r}, t)|^2} \quad (15)$$

We shall assume that, provided the time average is taken over a sufficiently long period of time for the scatterer configuration to pass through the majority of states in the statistical ensemble, the time average

of a physical quantity is essentially equal to the corresponding configurational average; for example:

$$\overline{\psi(\vec{r}, t)} \approx \langle \psi(\vec{r}) \rangle , \quad (16)$$

$$\overline{|\psi(\vec{r}, t)|^2} \approx \langle |\psi(\vec{r})|^2 \rangle . \quad (17)$$

With this assumption, which is analogous to the ergodic hypothesis of statistical mechanics, we may provide an alternative description of the configurational averages as being equal to the corresponding time averages for a single configuration.

## II. CONFIGURATIONAL AVERAGES OF EQUATIONS OF MULTIPLE SCATTERING

The self-consistent field equations of multiple scattering, completely accounting for the effect on each scatterer due to the combined presence of all the other scatterers in the configuration, has been given by Foldy<sup>1</sup> as

$$\psi(\vec{r}) = \psi_i(\vec{r}) + \sum_j g_j \psi^j(\vec{r}_j) E(\vec{r}, \vec{r}_j) , \quad (18)$$

$$\psi^j(\vec{r}) = \psi_i(\vec{r}) + \sum_{k \neq j} g_k \psi^k(\vec{r}_k) E(\vec{r}, \vec{r}_k) , \quad (19)$$

where we denote

$$g_j = g(\vec{r}_j, R_j) , \quad (20)$$

$$E(\vec{r}, \vec{r}_j) = \frac{e^{i\kappa_0 |\vec{r} - \vec{r}_j|}}{|\vec{r} - \vec{r}_j|} . \quad (21)$$

Equation (18) states that the total field may be expressed as the sum of

the incident wave and the spherically symmetric waves from each of the scatterers in the configuration; the latter gives the wave incident on the  $j$ 'th scatterer as the sum of the incident wave plus the waves from all the other scatterers. These equations are rigorous as they stand and include all orders of multiple scattering.

In order to obtain an integral equation satisfied by the average field  $\langle \psi(\vec{r}) \rangle$ , let us multiply each term of Eq. (18) by the joint probability distribution  $p(\vec{r}_1, R_1; \dots; \vec{r}_N, R_N)$  and integrate over all its coordinates. By Eq. (10) the left hand side of the resulting equation is then just the configurational average  $\langle \psi(\vec{r}) \rangle$ . Since the incident field  $\psi_i(\vec{r})$  is independent of the locations or radii of the scatterers, the second term is left unaltered. The third term may be evaluated by use of the conditional probability decomposition (7) and the definition (11) of the conditional average of  $\psi^j(\vec{r}_j)$ , with the result

$$\langle \psi(\vec{r}) \rangle = \psi_i(\vec{r}) + \frac{1}{N} \sum_j \int G(\vec{r}_j) \langle \psi^j(\vec{r}_j) \rangle_j E(\vec{r}, \vec{r}_j) d\vec{r}_j, \quad (22)$$

where the scatterer coefficient density is defined by

$$G(\vec{r}) = \int_0^\infty g(\vec{r}, R) n(\vec{r}, R) dR. \quad (23)$$

In order to simplify Eq. (22), the following approximation will be introduced

$$\langle \psi^j(\vec{r}_j) \rangle_j \approx \langle \psi(\vec{r}_j) \rangle; \quad (24)$$

that is, the configurational average of the exciting field of the  $j$ 'th scatterer, averaged over a statistical ensemble of configurations having the  $j$ 'th scatterer fixed, is approximately equal to the configurational

average of the total field at the same point. The validity of this assumption has been discussed by Lax<sup>8</sup>, and more recently by Waterman and Truell<sup>4</sup>. The error introduced would be expected to be  $O\left(\frac{1}{N}\right)$ ; therefore the assumption may be considered to be valid when the number  $N$  of scatterers is large. Since each of the  $N$  terms in the sum in Eq. (22) becomes identical, the governing integral equation for the configurational average of the complex wave function becomes:

$$\langle \psi(\vec{r}) \rangle = \psi_i(\vec{r}) + \int G(\vec{r}, \vec{r}') \langle \psi(\vec{r}') \rangle E(\vec{r}, \vec{r}') d\vec{r}' . \quad (25)$$

In a similar fashion Foldy derived an integral equation for  $\langle \psi(\vec{r}) \psi^*(\vec{r}_0) \rangle$ ; Foldy's result may be expressed as the representation:

$$\langle \psi(\vec{r}) \psi^*(\vec{r}_0) \rangle - \langle \psi(\vec{r}) \rangle \langle \psi(\vec{r}_0) \rangle^* = \frac{1}{4\pi} \int \Sigma_s(\vec{r}') \langle |\psi(\vec{r}')|^2 \rangle L(\vec{r}, \vec{r}_0; \vec{r}') d\vec{r}' , \quad (26)$$

where the kernel  $L(\vec{r}, \vec{r}_0; \vec{r}')$  satisfies the following integral equation

$$\begin{aligned} L(\vec{r}, \vec{r}_0; \vec{r}') &= E(\vec{r}, \vec{r}') E^*(\vec{r}_0, \vec{r}') \\ &+ \iint G(\vec{r}'') G^*(\vec{r}''') L(\vec{r}'', \vec{r}'''; \vec{r}') E(\vec{r}, \vec{r}'') E^*(\vec{r}_0, \vec{r}''') d\vec{r}'' d\vec{r}''' . \end{aligned} \quad (27)$$

It has been shown, however, that the above result is not a valid representation<sup>9, 10</sup>. Instead of considering Eq. (27) further, we shall first derive a conservation relation governing the average intensity and current. In the next section we shall use this relation in order to develop a consistent

<sup>8</sup>M. Lax, "Multiple Scattering of Waves II", Phys. Rev. 85, 621 - 629 (1952).

<sup>9</sup>D. J. McCloskey, "Multiple Scattering of Acoustical Waves", Ph.D. Thesis - California Institute of Technology, 1967.

<sup>10</sup>V. Twersky, "On Propagation in Random Media of Discrete Scatterers", Proc. Sym. Appl. Math. XVI, 84 - 116.

expression for the kernel  $L(\vec{r}, \vec{r}_0; \vec{r}')$ .

Let us now consider a given configuration of scatterers and construct an arbitrary surface  $S$  containing an arbitrary number of the scatterers as well as a sphere  $S_j$  of radius  $\rho_j$  about each scatterer  $j$ . Take each radius  $\rho_j$  sufficiently small such that none of these surfaces intersect and let  $V$  denote the multiply-connected volume enclosed between the surface  $S$  and the spheres  $S_j$  lying within  $S$ . The total field  $\psi(\vec{r})$  is regular throughout the volume  $V$ , including its boundaries; by applying the divergence theorem and using the wave equation (1), we may write

$$\begin{aligned} & \int_S [\psi^*(\vec{r}) \nabla \psi(\vec{r}) - \psi(\vec{r}) \nabla \psi^*(\vec{r})] \cdot d\vec{S} \\ &= \sum_j \gamma_j \int_{S_j} [\psi^*(\vec{r}) \nabla \psi(\vec{r}) - \psi(\vec{r}) \nabla \psi^*(\vec{r})] \cdot d\vec{S}_j \\ & \quad - \int_V (\kappa_o^2 - \kappa_o^{*2}) |\psi(\vec{r})|^2 d\vec{r} \quad , \end{aligned} \quad (28)$$

where  $\gamma_j$  is equal to unity if the  $j$ 'th scatterer lies within  $S$ , and zero if it lies outside  $S$ . The total field  $\psi(\vec{r})$  is described by the fundamental equations (18) and (19) of multiple scattering; in the neighborhood of the  $j$ 'th scatterer, it may be expressed as the sum of the exciting field  $\psi^j(\vec{r})$  and the scattered wave,

$$\psi(\vec{r}) = \psi^j(\vec{r}) + g_j \psi^j(\vec{r}_j) E(\vec{r}, \vec{r}_j) \quad . \quad (29)$$

After substituting this result into Eq. (28) and letting the radius  $\rho_j$  of each of the spherical surfaces  $S_j$  tend to zero, the following result is obtained



$$\begin{aligned}
& \int_S [\psi^*(\vec{r}) \nabla \psi(\vec{r}) - \psi(\vec{r}) \nabla \psi^*(\vec{r})] \cdot d\vec{S} \\
& = -2ik_0 \left\{ \sum_j \gamma_j \sigma_a^j |\psi^j(\vec{r}_j)|^2 + \int_V \alpha_0 |\psi(\vec{r})|^2 d\vec{r} \right\} , \quad (30)
\end{aligned}$$

where the absorption cross section of the  $j$ 'th scatterer is given by

$$\sigma_a^j = \frac{4\pi}{k_0} \text{Im}\{g_j\} - 4\pi |g_j|^2 . \quad (31)$$

Equation (30) represents a simple statement of conservation of energy; it states that the mean energy flux through the surface  $S$  is equal to the rate of energy absorption by the scatterers plus the rate of energy dissipation in the matrix medium.

In order to obtain a conservation relation for a random distribution of scatterers, the configurational average may be taken of Eq. (30) by multiplying each term by the joint probability distribution  $p(\vec{r}_1, R_1; \dots; \vec{r}_N, R_N)$  and integrating over the coordinates of all the scatterers; this results in the following integral relation for the configurational averages

$$\begin{aligned}
& \int_S \langle \psi^*(\vec{r}) \nabla \psi(\vec{r}) - \psi(\vec{r}) \nabla \psi^*(\vec{r}) \rangle \cdot d\vec{S} \\
& = -2ik_0 \int_V \Sigma_a(\vec{r}) \langle |\psi(\vec{r})|^2 \rangle d\vec{r} , \quad (32)
\end{aligned}$$

where the absorption cross section density is the difference of the extinction and scattering cross section densities of Eqs. (4) and (5).

### III. COHERENT AND INCOHERENT SCATTERING

In the previous section we have obtained governing equations (25)

and (26) for  $\langle \psi(\vec{r}) \rangle$  and  $\langle \psi(\vec{r}) \psi^*(\vec{r}_0) \rangle$ , as well as an integral relation (32) connecting  $\langle \psi^*(\vec{r}) \nabla \psi(\vec{r}) - \psi(\vec{r}) \nabla \psi^*(\vec{r}) \rangle$  and  $\langle |\psi(\vec{r})|^2 \rangle$ . We shall now examine the physical consequence of these equations and reduce them to more workable forms.

#### A. The Coherent Wave

By operating on Eq. (25) with  $(\nabla^2 + \kappa_0^2)$ , Foldy showed that the average wave satisfies the wave equation

$$[\nabla^2 + \kappa^2(\vec{r})] \langle \psi(\vec{r}) \rangle = 0, \quad (33)$$

where the complex propagation coefficient  $\kappa(\vec{r})$  of the scattering medium is given by

$$\kappa(\vec{r}) = k(\vec{r}) + i \frac{\alpha(\vec{r})}{2} = [\kappa_0^2 + 4\pi G(\vec{r})]^{\frac{1}{2}}. \quad (34)$$

Hence the physical behavior of the coherent wave is quite simple. According to Eq. (33), the coherent wave satisfies a wave equation having a complex propagation coefficient that depends, in general, on position. Thus the incident wave and the scattered waves from all the scatterers interfere, on the average, to form a new wave travelling at a different phase velocity and undergoing attenuation. This wave will display the reflection and refraction aspects of coherent scattering at surfaces of discontinuity.

#### B. Conservation of Coherent and Incoherent Sound Energy

Let us return to the integral relation (32) connecting  $\langle \psi^*(\vec{r}) \nabla \psi(\vec{r}) - \psi(\vec{r}) \nabla \psi^*(\vec{r}) \rangle$  and  $\langle |\psi(\vec{r})|^2 \rangle$ . Since the configurational average of the current is regular on and within the surface  $S$ , the divergence

theorem may be employed in order to rewrite the relation in differential form as

$$\operatorname{div} \langle \psi^*(\vec{r}) \nabla \psi(\vec{r}) - \psi(\vec{r}) \nabla \psi^*(\vec{r}) \rangle = -2ik_0 \Sigma_a(\vec{r}) \langle |\psi(\vec{r})|^2 \rangle \quad (35)$$

A similar result may be obtained for the coherent wave, which satisfies the wave equation (33); using the definition (34) of the propagation constant of the scattering medium and the definition (5) of the extinction cross section density we obtain

$$\operatorname{div} [\langle \psi(\vec{r}) \rangle^* \nabla \langle \psi(\vec{r}) \rangle - \langle \psi(\vec{r}) \rangle \nabla \langle \psi(\vec{r}) \rangle^*] = -2ik_0 \Sigma_e(\vec{r}) |\langle \psi(\vec{r}) \rangle|^2 \quad (36)$$

In order to understand the physical significance of Eqs. (35) and (36), let us denote the configurational average of the total sound intensity and sound energy current by

$$e(\vec{r}) = \frac{\rho_0 \omega^2}{2c_0^2} \langle |\psi(\vec{r})|^2 \rangle \quad , \quad (37)$$

$$\vec{j}(\vec{r}) = \frac{\rho_0 \omega}{4i} \langle \psi^*(\vec{r}) \nabla \psi(\vec{r}) - \psi(\vec{r}) \nabla \psi^*(\vec{r}) \rangle \quad , \quad (38)$$

where  $\rho_0$  and  $c_0$  denote the density and phase velocity of the matrix medium and  $\omega$  is the frequency of the harmonic sound field. The "coherent" contributions to the average sound intensity and average sound energy current may be defined by

$$e_c(\vec{r}) = \frac{\rho_0 \omega^2}{2c_0^2} |\langle \psi(\vec{r}) \rangle|^2 \quad , \quad (39)$$

$$\vec{j}_c(\vec{r}) = \frac{\rho_0 \omega}{4i} [\langle \psi(\vec{r}) \rangle^* \nabla \langle \psi(\vec{r}) \rangle - \langle \psi(\vec{r}) \rangle \nabla \langle \psi(\vec{r}) \rangle^*] \quad , \quad (40)$$

with the "incoherent" contributions to these quantities being defined as the differences of the total and coherent quantities. Equations (35) and

(36) may be rewritten, in terms of these definitions, as follows:

$$\text{div } \vec{j}(\vec{r}) = -c_o \Sigma_a(\vec{r}) e(\vec{r}) \quad , \quad (41)$$

$$\text{div } \vec{j}_c(\vec{r}) = -c_o \Sigma_e(\vec{r}) e_c(\vec{r}) \quad . \quad (42)$$

The first of these is a statement of conservation of total sound energy; the latter describes the conservation of the coherent portion. Upon subtraction, we obtain a statement of conservation of the incoherent portion of the sound energy,

$$\text{div } \vec{j}_i(\vec{r}) = c_o \Sigma_s(\vec{r}) e_c(\vec{r}) - c_o \Sigma_a(\vec{r}) e_i(\vec{r}) \quad . \quad (43)$$

This last equation now shows the essential connection between the coherent and incoherent contributions to the sound intensity and current. By application of the divergence theorem to the above equations, their physical consequences becomes apparent. It is seen that the flux of coherent sound energy across an arbitrary surface  $S$  is equal to the combined rates of absorption and scattering within  $S$ . The coherent energy lost as a result of scattering appears as a source of incoherent energy, as evidenced by its presence on the right hand side of Eq. (43). This equation states that the flux of incoherent sound energy across  $S$  is equal to the rate of production of incoherent sound energy within  $S$  by the scattering of the coherent energy minus the rate of absorption of incoherent sound energy within  $S$ .

### C. Incoherent Scattering

Let us now return to the governing integral equation (26) for  $\langle \psi(\vec{r}) \psi^*(\vec{r}_o) \rangle - \langle \psi(\vec{r}) \rangle \langle \psi(\vec{r}_o) \rangle^*$ , from which the incoherent contributions to the intensity and current may be determined. As has been pointed out,

Foldy's representation (27) is not a valid description of the kernel  $L(\vec{r}, \vec{r}_0; \vec{r}')$ ; in order to provide a consistent formulation, we shall determine the expression for  $L(\vec{r}, \vec{r}_0; \vec{r}')$  by requiring the representation (26) to be consistent with the conservation relations (35) and (36) of the previous section. If we differentiate Eq. (26) first with respect to  $\vec{r}$ , then with respect to  $\vec{r}_0$ , subtract, set  $\vec{r}_0 = \vec{r}$ , and then take the divergence of each term of the resulting equation, we obtain

$$\begin{aligned} & \text{div} \langle \psi^*(\vec{r}) \nabla \psi(\vec{r}) - \psi(\vec{r}) \nabla \psi^*(\vec{r}) \rangle - \text{div} [ \langle \psi(\vec{r}) \rangle^* \nabla \langle \psi(\vec{r}) \rangle - \langle \psi(\vec{r}) \rangle \nabla \langle \psi(\vec{r}) \rangle^* ] \\ &= \frac{1}{4\pi} \int \Sigma_s(\vec{r}') \langle |\psi(\vec{r}')|^2 \rangle \nabla \cdot [ (\nabla L - \nabla_0 L)(\vec{r}, \vec{r}; \vec{r}') ] d\vec{r}' \quad . \end{aligned} \quad (44)$$

By employing the conservation relations (35) and (36) to eliminate these divergences, it may be shown that the kernel  $L(\vec{r}, \vec{r}_0; \vec{r}')$  of Eq. (26) satisfies the following relation:

$$\nabla \cdot [ (\nabla L - \nabla_0 L)(\vec{r}, \vec{r}; \vec{r}') ] = 8\pi i k_0 \delta(\vec{r} - \vec{r}') - 2i k_0 \Sigma_e(\vec{r}) L(\vec{r}, \vec{r}; \vec{r}') \quad , \quad (45)$$

where  $\delta(\vec{r})$  is the three-dimensional delta function. If we take  $L(\vec{r}, \vec{r}_0; \vec{r}')$  to be a product of a function of  $\vec{r}$  times a function of  $\vec{r}_0$ , this relation may be written as

$$\{ [ \nabla^2 + \kappa^2(\vec{r}) ] L(\vec{r}, \vec{r}_0; \vec{r}') - [ \nabla_0^2 + \kappa^{*2}(\vec{r}) ] L(\vec{r}, \vec{r}_0; \vec{r}') \}_{\vec{r}=\vec{r}_0} = 8\pi i k_0 \delta(\vec{r} - \vec{r}') \quad . \quad (46)$$

One may readily verify that a solution to this equation is

$$L(\vec{r}, \vec{r}_0; \vec{r}') = \gamma(\vec{r}') K(\vec{r}, \vec{r}') K^*(\vec{r}_0, \vec{r}') \quad , \quad (47)$$

where  $K(\vec{r}, \vec{r}')$  is the outgoing solution of the coherent wave equation,

$$[\nabla^2 + \kappa^2(\vec{r})]K(\vec{r}, \vec{r}') = -4\pi\delta(\vec{r} - \vec{r}') \quad (48)$$

and

$$\gamma(\vec{r}) = \frac{k_0}{\text{Im}\{K(\vec{r}, \vec{r}')\} \big|_{\vec{r}=\vec{r}'}} \quad (49)$$

The incoherent contribution to the mean square wave is obtained as a special case of Eq. (26) by setting  $\vec{r}_0 = \vec{r}$  yielding

$$\langle |\psi(\vec{r})|^2 \rangle - |\langle \psi(\vec{r}) \rangle|^2 = \frac{1}{4\pi} \int \gamma(\vec{r}') \Sigma_s(\vec{r}) \langle |\psi(\vec{r}')|^2 \rangle |K(\vec{r}, \vec{r}')|^2 d\vec{r}' ; \quad (50)$$

the incoherent contribution to the average current is obtained from Eq. (26) by first differentiating with respect to  $\vec{r}$ , then with respect to  $\vec{r}_0$ , subtracting, and setting  $\vec{r}_0 = \vec{r}$ , with the result

$$\begin{aligned} & \langle \psi^*(\vec{r}) \nabla \psi(\vec{r}) - \psi(\vec{r}) \nabla \psi^*(\vec{r}) \rangle - [\langle \psi(\vec{r}) \rangle^* \nabla \langle \psi(\vec{r}) \rangle - \langle \psi(\vec{r}) \rangle \nabla \langle \psi(\vec{r}) \rangle^*] \\ &= \frac{1}{4\pi} \int \gamma(\vec{r}') \Sigma_s(\vec{r}') \langle |\psi(\vec{r}')|^2 \rangle [K^*(\vec{r}, \vec{r}') \nabla K(\vec{r}, \vec{r}') - K(\vec{r}, \vec{r}') \nabla K^*(\vec{r}, \vec{r}')] d\vec{r}' . \end{aligned} \quad (51)$$

#### IV. WAVE PROPAGATION IN AN INFINITE SCATTERING MEDIUM

In order to illustrate the rather general theory presented heretofore, let us now consider the special case in which the entire space is filled by a random distribution of scatterers; assume that the scatterer number density  $n(R)$  and the scatterer coefficient  $g(R)$  of a single scatterer are independent of position so that the scatterer coefficient density

$$G = \int_0^\infty g(R)n(R)dR \quad , \quad (52)$$

propagation constant of the scattering medium

$$\kappa = k + i \frac{\alpha}{2} = (\kappa_0^2 + 4\pi G)^{\frac{1}{2}} \quad , \quad (53)$$

as well as the scattering and extinction cross section densities

$$\Sigma_s = 4\pi \int_0^\infty |g(R)|^2 n(R)dR \quad , \quad (54)$$

$$\Sigma_e = \frac{4\pi}{k_0} \int_0^\infty \text{Im}\{g(R)\} n(R)dR + \alpha_0 \quad , \quad (55)$$

are all constant. We note that the scattering coefficient of a single scatterer, along with the scatterer number density distribution, completely determines the macroscopic properties of the scattering medium.

By Eq. (33), the coherent wave  $\langle \psi(\vec{r}) \rangle$  satisfies the wave equation

$$(\nabla^2 + \kappa^2) \langle \psi(\vec{r}) \rangle = 0 \quad ; \quad (56)$$

the average pressure, coherent contribution to the average sound intensity, and coherent contribution to the average sound energy current are related to this quantity by the following formulas:

$$p(\vec{r}, t) = \text{Re}\{i\omega\rho_0 \langle \psi(\vec{r}) \rangle e^{-i\omega t}\} \quad , \quad (57)$$

$$e_c(\vec{r}) = \frac{\rho_0 \omega^2}{2c_0^2} |\langle \psi(\vec{r}) \rangle|^2 \quad (58)$$

$$j_c(\vec{r}) = \frac{\rho_0 \omega}{4i} [\langle \psi(\vec{r}) \rangle^* \nabla \langle \psi(\vec{r}) \rangle - \langle \psi(\vec{r}) \rangle \nabla \langle \psi(\vec{r}) \rangle^*] \quad . \quad (59)$$

Therefore the properties of the coherent radiation are completely



determined by the solution of the wave equation (56).

For the case of constant propagation coefficient, Eq. (48) has the solution

$$K(\vec{r}, \vec{r}') = \frac{e^{iK|\vec{r}-\vec{r}'|}}{|\vec{r}-\vec{r}'|} \quad (60)$$

so that  $\gamma(\vec{r}')$ , defined by Eq. (49), is given by

$$\gamma(\vec{r}') = \frac{k_o}{k} = \frac{\alpha}{\Sigma_e} \quad (61)$$

By multiplying Eq. (50) by  $\frac{\rho_o \omega^2}{2c_o^2}$  and substituting these expressions for  $K(\vec{r}, \vec{r}')$  and  $\gamma(\vec{r}')$ , we obtain the following governing equation for the incoherent contribution to the average sound intensity

$$e_i(\vec{\tau}) = \beta \int [e_c(\vec{\tau}') + e_i(\vec{\tau}')] \frac{e^{-|\vec{\tau}-\vec{\tau}'|}}{4\pi|\vec{\tau}-\vec{\tau}'|^2} d\vec{\tau}' \quad (62)$$

where distances are expressed in terms of the dimensionless variable  $\vec{\tau} = \alpha \vec{r}$  and

$$\beta = \frac{\Sigma_s}{\Sigma_e} \quad (63)$$

Here we have the very interesting result, that for this special case in which the scatterer density is uniform throughout all space, the governing equation for the incoherent contribution to the average sound intensity is identical in form and physical interpretation to the Boltzmann integral equation describing the transport of monoenergetic neutrons in an infinite homogeneous medium. That is, we need only to replace  $e_i(\vec{\tau})$  by the neutron density  $n(\vec{\tau})$  and  $\beta e_c(\vec{\tau})$  by the neutron source density  $S(\vec{\tau})$  in order to obtain the fundamental equation of neutron transport;  $\beta$  retains the same significance as the ratio of scattering to total macroscopic cross section. Thus it is seen that the incoherent radiation is governed by an equation displaying "particle" aspects, as opposed to the coherent radiation which is governed by a "wave" equation. It should be noted that in the preceding general

theory no assumption has been made that the wavelength is small compared with the average distance between scatterers. Therefore, there is no reason to assume à priori that the incoherent radiation satisfies the transport equation for particles, since there are no wave "packets" which may be treated as independent discrete entities. Indeed, this strict particle analogy results only for this one special case in which there are no discontinuities in the average scattering characteristics of the medium, with a resulting absence of any specular reflection or refraction.

By substituting the expression (60) for  $K(\vec{r}, \vec{r}')$  into Eq. (51) and using the definitions (37) and (38), we obtain the following expression for the incoherent contribution to the average sound energy current:

$$\vec{j}(\vec{r}) = \int c_o \Sigma_s [e_c(\vec{r}') + e_i(\vec{r}')] \frac{(\vec{r} - \vec{r}')}{4\pi |\vec{r} - \vec{r}'|^3} e^{-\alpha |\vec{r} - \vec{r}'|} d\vec{r}' . \quad (64)$$

The physical consequence of this equation is also easily understood since it is an exact dual of the corresponding equation for particles. Here  $c_o \Sigma_s [e_c(\vec{r}') + e_i(\vec{r}')] d\vec{r}'$  is the rate that incoherent radiation is scattered away from  $d\vec{r}'$  (recall from Eq. (43) that  $c_o \Sigma_s e_c(\vec{r}')$  is the rate of production of incoherent sound energy per unit volume due to the scattering of the coherent wave). This, multiplied by the kernel, gives the average current at  $\vec{r}$  due to incoherent radiation scattered from  $d\vec{r}'$ ; integration over all space then yields the total incoherent contribution to the average current. Again, it is only for this special case that such a "particle" interpretation is possible.

The average sound intensity, which is proportional to the average

mean square pressure perturbation is obtained by summing the coherent and incoherent contributions, as follows:

$$e(\vec{r}) = \frac{1}{\rho_o c_o^2} \left\langle \overline{p(\vec{r}, t)^2} \right\rangle = e_c(\vec{r}) + e_i(\vec{r}) \quad . \quad (65)$$

The average sound energy current is equal to the average of the mean value of the product of the fluid pressure perturbation and velocity; it may be computed in a similar manner

$$\vec{j}(\vec{r}) = \left\langle \overline{p(\vec{r}, t) \vec{V}(\vec{r}, t)} \right\rangle = \vec{j}_c(\vec{r}) + \vec{j}_i(\vec{r}) \quad . \quad (66)$$

If we assume that our previous "ergodic" hypothesis applies, the above configurational averages of mean quantities are equal to the time averages of the quantities, taken over a sufficiently long period of time.

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1. ORIGINATING ACTIVITY (Corporate author) California Institute of Technology Division of Engineering and Applied Science		2a. REPORT SECURITY CLASSIFICATION Unclassified	
		2b. GROUP Not Applicable	
3. REPORT TITLE  Multiple Scattering of Acoustical Waves			
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5. AUTHOR(S) (Last name, first name, initial)  McCloskey, David J.			
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10. AVAILABILITY/LIMITATION NOTICES  DISTRIBUTION OF THIS DOCUMENT IS UNLIMITED			
11. SUPPLEMENTARY NOTES		12. SPONSORING MILITARY ACTIVITY  Office of Naval Research	
13. ABSTRACT  The general theory of the multiple scattering of acoustical waves by a random distribution of isotropic point scatterers is considered. Configurational averages are taken of the equations of multiple scattering and integral equations governing these configurational averages are obtained; the physical consequences of these equations are examined in detail. A complete theoretical picture is obtained of the propagation of the coherent and incoherent radiation and the connection between the coherent and incoherent contributions to the average sound intensity and current.			

14. KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
Acoustical waves						
Multiple scattering						

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